On the Sobolev and Hardy constants for the fractional Navier Laplacian

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Abstract. We prove the coincidence of the Sobolev and Hardy constants relative to the "Dirichlet" and "Navier" fractional Laplacians of any real order $m \in (0, \frac{n}{2})$ over bounded domains in $\mathbb{R}^n$.

1 Introduction

For any integer $n \geq 1$ the (fractional) Laplacian of real order $m > 0$ over $\mathbb{R}^n$ is defined by

$$\mathcal{F}\left[(-\Delta)^m_D u\right] = |\xi|^{2m}\mathcal{F}[u]\,,$$

where $\mathcal{F}$ is the Fourier transform

$$\mathcal{F}[u](\xi) = \frac{1}{(2\pi)^{n/2}}\int\limits_{\mathbb{R}^n} e^{-i\xi\cdot x}u(x)\,dx\,.$$

Let $p \in (1,\infty)$ and assume $n > pm$. Put $I_m(f) = |x|^{m-n} \star f$. Then the Hardy–Littlewood–Sobolev inequality [9, 10, 18] states that I_m is continuous operator from $L^p(\mathbb{R}^n)$ to $L^{p^*_m}(\mathbb{R}^n)$, where

$$p^*_m := \frac{pn}{n-pm}$$

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is the *critical Sobolev exponent.*

We denote by $\mathcal{D}^{m,p}(\mathbb{R}^n)$ the image of I_m. Since for any $f \in L^p(\mathbb{R}^n)$

$$(-\Delta)_D^{\frac{m}{2}}\big(|x|^{m-n} \star f\big) = c_{n,m} \cdot f$$

in the distributional sense on $\mathbb{R}^n$ (here the constant $c_{n,m}$ depends only on n and m), we have

$$\mathcal{D}^{m,p}(\mathbb{R}^n) = \{u \in L^{p_m^*}(\mathbb{R}^n) \mid (-\Delta)_D^{\frac{m}{2}} u \in L^p(\mathbb{R}^n) \}.$$

We endow $\mathcal{D}^{m,p}(\mathbb{R}^n)$ with the norm

$$\|u\|_{\mathcal{D}^{m,p}} = \| (-\Delta)_D^{\frac{m}{2}} u\|_p := \Big(\int\limits_{\mathbb{R}^n} | (-\Delta)_D^{\frac{m}{2}} u|^p \, dx\Big)^{1/p},$$

so that $I_m : L^p(\mathbb{R}^n) \to \mathcal{D}^{m,p}(\mathbb{R}^n)$ is (up to a constant) an isometry with inverse $(-\Delta)_D^{\frac{m}{2}}$. In particular, $\mathcal{D}^{m,p}(\mathbb{R}^n)$ is a reflexive Banach space.

In the Hilbertian case $p = 2$ we will simply write $\mathcal{D}^m(\mathbb{R}^n)$ instead of $\mathcal{D}^{m,2}(\mathbb{R}^n)$. The explicit value and the extremals of the best constant $\mathcal{S}_m$ in the inequality

$$\int\limits_{\mathbb{R}^n} | (-\Delta)_D^{\frac{m}{2}} u|^2 \, dx \geq \mathcal{S}_m \Big(\int\limits_{\mathbb{R}^n} |u|^{2_m^*} \, dx\Big)^{\frac{2}{2_m^*}} \quad \text{for any } u \in \mathcal{D}^m(\mathbb{R}^n)$$

were furnished by Cotsiolis and Tavoularis in [4].

Next, we introduce the "Dirichlet" Laplacian of order m over a bounded and smooth domain $\Omega \subset \mathbb{R}^n$ via the quadratic form

$$Q_m^D[u] = ((-\Delta)_D^m u, u) := \int\limits_{\mathbb{R}^n} | (-\Delta)_D^{\frac{m}{2}} u|^2 \, dx \,,$$

with domain

$$\widetilde{H}^m(\Omega) = \{u \in \mathcal{D}^m(\mathbb{R}^n) \, : \, \operatorname{supp} u \subset \overline{\Omega}\} \,.$$

We endow $\widetilde{H}^m(\Omega)$ with the norm $\| \cdot \|_{\mathcal{D}^m}$. Since $\mathcal{C}_0^\infty$ is dense in $\mathcal{D}^m(\mathbb{R}^n)$, a standard dilation argument implies that

$$\mathcal{S}_m = \inf_{\substack{u \in \widetilde{H}^m(\Omega) \\ u \neq 0}} \frac{Q_m^D[u]}{\|u\|_{2_m^*}^2} \,.$$

We introduce also the "Navier" Laplacian $(-\Delta)^m_N$ of order m over Ω as the m^{th} power of the conventional Laplacian $-\Delta$ on $H^1_0(\Omega)$, in the sense of spectral theory. More precisely, for $u \in L^2(\Omega)$ we define

$$(-\Delta)^m_N u := \sum_{j\ge 1} \lambda_j^m \Big(\int\limits_\Omega u\varphi_j\, dx \Big) \varphi_j .$$

Here λ_j, φ_j are, respectively, the eigenvalues and eigenfunctions (normalized in $L^2(\Omega)$) of $-\Delta$ on $H^1_0(\Omega)$ while the series converges in the sense of distributions.

The corresponding quadratic form is

$$Q^N_m[u] = ((-\Delta)^m_N u, u) = \sum_{j\ge 1} \lambda_j^m \Big(\int\limits_\Omega u\varphi_j\, dx \Big)^2 = \int\limits_\Omega |\, (-\Delta)^{\frac{m}{2}}_N u|^2\, dx,$$

with domain

$$\widetilde{H}^m_N(\Omega) = \{u \in L^2(\Omega)\ :\ Q^N_m[u] < \infty\}.$$

Finally, we define the *Navier-Sobolev* constant by

$$\mathcal{S}^N_m := \inf_{\substack{u\in \widetilde{H}^m_N(\Omega)\\ u\ne 0}} \frac{Q^N_m[u]}{\|u\|^2_{2^*_m}} .$$

We are in position to state the main result of the present paper.

Theorem 1 *Let Ω be a bounded and smooth domain in $\mathbb{R}^n$ and $m \in \big(0, \frac{n}{2}\big)$. Then*

$$\mathcal{S}^N_m = \mathcal{S}_m .$$

Our argument applies also to Hardy-Rellich type inequalities. The explicit value of the positive constant

$$\mathcal{H}_m := \inf_{\substack{u\in \mathcal{D}^m(\mathbb{R}^n)\\ U\ne 0}} \frac{Q^D_m[u]}{\||x|^{-m}u\|^2_2} = \inf_{\substack{u\in \widetilde{H}^m(\Omega)\\ U\ne 0}} \frac{Q^D_m[u]}{\||x|^{-m}u\|^2_2}$$

has been computed in [11] (see also [5] and [13] for the integer orders $m \in \mathbb{N}$, even in a non-Hilbertian setting). The *Navier-Hardy* constant over a bounded and smooth domain Ω is defined by

$$\mathcal{H}^N_m := \inf_{\substack{u\in \widetilde{H}^m_N(\Omega)\\ u\ne 0}} \frac{Q^N_m[u]}{\||x|^{-m}u\|^2_2} .$$

The argument we use to prove Theorem 1 plainly leads to the next result.

Theorem 2 *Let Ω be a bounded and smooth domain in $\mathbb{R}^n$ and $m \in \left(0, \frac{n}{2}\right)$. Then*

$$\mathcal{H}_m^N = \mathcal{H}_m.$$

The equalities $\mathcal{S}_1^N = \mathcal{S}_1$, $\mathcal{H}_1^N = \mathcal{H}_1$ are totally trivial. If $m \neq 1$ is an integer number, then the inequalities $\mathcal{S}_m^N \leq \mathcal{S}_m$ and $\mathcal{H}_m^N \leq \mathcal{H}_m$ follow immediately from $\widetilde{H}^m(\Omega) \subseteq \widetilde{H}_N^m(\Omega)$, whereas the opposite inequalities need a detailed proof.

For integer orders $m \in \mathbb{N}$, the statements of Theorems 1 and 2 are known (even in non-Hilbertian setting). The coincidence of the two Hardy constants can be extracted from the proof of Theorem 3.3 in [13] (see also [6, Lemma 1]), where Enzo Mitidieri took advantage of a Rellich–Pokhozhaev type identity [17, 12]. The coincidence of the two Sobolev constants for $m \in \mathbb{N}$ was obtained in [7] (see also [8, 21] for previous results in case $p = 2$ and $m = 2$). We cite also [14], where weighted Sobolev constants are studied under the hypothesis $m = 2$.

We emphasize that for $m \notin \mathbb{N}$ none of the inequalities $\mathcal{S}_m^N \leq \mathcal{S}_m$, $\mathcal{S}_m^N \geq \mathcal{S}_m$ (respectively, $\mathcal{H}_m^N \leq \mathcal{H}_m$, $\mathcal{H}_m^N \geq \mathcal{H}_m$) is easily checked. For $m \in (0, 1)$, Theorem 1 was proved in [15]. To handle the general case of real orders $m > 0$ we largely use some of the results in [15, 16]. Additional tools are the maximum principles for fractional Laplacians and a result about the transform $u \mapsto |u|$, $u \in \widetilde{H}^m(\Omega)$, for $0 < m < 1$, that might have an independent interest (see Theorem 3).

2 Preliminaries

Here we collect some facts about the Dirichlet and the Navier quadratic forms.

1. First, we note that $\widetilde{H}^m(\Omega) \subseteq \widetilde{H}_N^m(\Omega)$ and

$$\widetilde{H}^m(\Omega) = \widetilde{H}_N^m(\Omega) \quad \text{if and only if} \quad m < \frac{3}{2}.$$

This fact is well known for natural orders m; the general case follows immediately from [20, Theorem 1.17.1/1] and [20, Theorem 4.3.2/1].

2. It is well known that for any $m \in \mathbb{N}$

$$\widetilde{H}_N^m(\Omega) = \left\{ u \in H^m(\Omega) \;\middle|\; \mathrm{tr}_{\partial\Omega}\left[(-\Delta)^\nu u\right] = 0 \;\text{ for }\; \nu \in \mathbb{N}_0,\ \nu < \frac{m}{2} \right\}.$$

We omit the proof of the next simple analog for non integer m.

Lemma 1 *Let $m \notin \mathbb{N}$, $m > 1$.*

- *If $\lfloor m \rfloor \geq 2$ is even, then* $\widetilde{H}_N^m(\Omega) = \left\{ u \in \widetilde{H}_N^{\lfloor m \rfloor}(\Omega) \;\middle|\; (-\Delta)_N^{\frac{\lfloor m \rfloor}{2}} u \in \widetilde{H}^{m-\lfloor m \rfloor}(\Omega) \right\}$.
- *If $\lfloor m \rfloor \geq 1$ is odd, then* $\widetilde{H}_N^m(\Omega) = \left\{ u \in \widetilde{H}_N^{\lfloor m \rfloor}(\Omega) \;\middle|\; (-\Delta)_N^{\frac{m}{2}} u \in L^2(\Omega) \right\}$.

3. Let $m \in \mathbb{N}$ and let $u \in \widetilde{H}^m(\Omega)$. Then it is easy to see that $Q_m^D[u] = Q_m^N[u]$. More precisely, if m is even one gets the pointwise equality

$$(-\Delta)_D^{\frac{m}{2}} u = (-\Delta)_N^{\frac{m}{2}} u = (-\Delta)^{\frac{m}{2}} u.$$

If m is odd the following integral equalities hold:

$$\int\limits_{\mathbb{R}^n} |(-\Delta)_D^{\frac{m}{2}} u|^2 \, dx = \int\limits_{\Omega} |(-\Delta)_N^{\frac{m}{2}} u|^2 \, dx = \int\limits_{\Omega} |\nabla(\Delta^{\frac{m-1}{2}} u)|^2 \, dx.$$

Integrating by parts we can write for all $m \in \mathbb{N}$

$$\int\limits_{\mathbb{R}^n} |(-\Delta)_D^{\frac{m}{2}} u|^2 \, dx = \int\limits_{\Omega} |(-\Delta)_N^{\frac{m}{2}} u|^2 \, dx = \int\limits_{\Omega} |\nabla^m u|^2 \, dx, \qquad u \in \widetilde{H}^m(\Omega). \tag{2.1}$$

For non integer orders m the Dirichlet and Navier quadratic forms never coincide on the Dirichlet domain $\widetilde{H}^m(\Omega)$. Indeed, the next result holds.

Proposition 1 ([15, 16]) *Let $m > 0$, $m \notin \mathbb{N}$, and let $u \in \widetilde{H}^m(\Omega)$, $u \not\equiv 0$. Then*

$$\begin{aligned} \int\limits_{\mathbb{R}^n} |(-\Delta)_D^{\frac{m}{2}} u|^2 \quad &< \quad \int\limits_{\Omega} |(-\Delta)_N^{\frac{m}{2}} u|^2 \, dx \quad \text{if } \lfloor m \rfloor \text{ is even;} \\ \int\limits_{\mathbb{R}^n} |(-\Delta)_D^{\frac{m}{2}} u|^2 \quad &> \quad \int\limits_{\Omega} |(-\Delta)_N^{\frac{m}{2}} u|^2 \, dx \quad \text{if } \lfloor m \rfloor \text{ is odd.} \end{aligned}$$

In view of Proposition 1, one is lead to ask "how much" the Dirichlet and Navier quadratic forms differ on $\widetilde{H}^m(\Omega)$ if $m \notin \mathbb{N}$. The answer takes into account the action of dilations.

Fix any point $x_0 \in \Omega$ and take $u \in \widetilde{H}^m(\Omega)$. Concentrate u around x_0 by putting $u_\rho(x) = \rho^{\frac{n-2m}{2}} u(\rho(x - x_0) + x_0)$ for $\rho \gg 1$. Then $u_\rho \in \widetilde{H}^m(\Omega)$ and $Q_m^D[u_\rho] \equiv Q_m^D[u]$. In contrast, $Q_m^N[u_\rho]$ depends on ρ, as the Navier quadratic form does depend on the domain Ω. Nevertheless, the next result holds.

Proposition 2 ([15, 16]) *Let $m > 0$ and $u \in \widetilde{H}^m(\Omega)$. Then*

$$\int\limits_{\mathbb{R}^n} |\,(-\Delta)_D^{\frac{m}{2}} u|^2\, dx = \lim_{\rho\to\infty} \int\limits_{\Omega} |\,(-\Delta)_N^{\frac{m}{2}} u_\rho|^2\, dx.$$

4. It is well known that if $u \in \widetilde{H}^1(\Omega) = \widetilde{H}^1_N(\Omega) = H^1_0(\Omega)$ then $|u| \in \widetilde{H}^1(\Omega)$, and $|\nabla|u|| = |\nabla u|$ almost everywhere on Ω. By (2.1), this implies

$$\int\limits_{\mathbb{R}^n} |\,(-\Delta)_D^{\frac{1}{2}}|u||^2\, dx = \int\limits_{\mathbb{R}^n} |\,(-\Delta)_D^{\frac{1}{2}}u|^2\, dx = \int\limits_{\Omega} |\,(-\Delta)_N^{\frac{1}{2}}|u||^2\, dx = \int\limits_{\Omega} |\,(-\Delta)_N^{\frac{1}{2}}u|^2\, dx\,.$$

For smaller orders $m \in (0, 1)$ one still has that $\widetilde{H}^m(\Omega) = \widetilde{H}^m_N(\Omega)$ (see point 1 above), but the operator $u \mapsto |u|$ behaves quite differently.

Theorem 3 *Let $m \in (0, 1)$ and $u \in \widetilde{H}^m(\Omega)$. Then $|u| \in \widetilde{H}^m(\Omega)$ and*

$$\int\limits_{\mathbb{R}^n} |\,(-\Delta)_D^{\frac{m}{2}}|u||^2\, dx \;\leq\; \int\limits_{\mathbb{R}^n} |\,(-\Delta)_D^{\frac{m}{2}}u|^2\, dx \tag{2.2}$$

$$\int\limits_{\Omega} |\,(-\Delta)_N^{\frac{m}{2}}|u||^2\, dx \;\leq\; \int\limits_{\Omega} |\,(-\Delta)_N^{\frac{m}{2}}u|^2\, dx\,. \tag{2.3}$$

In addition, if both the positive and the negative parts of u are nontrivial, then strict inequalities hold in (2.2) and in (2.3).

Proof. In the paper [2], the Dirichlet fractional Laplacian of order $m \in (0, 1)$ was connected with the so-called *harmonic extension in $n+2-2m$ dimensions* (see also [1] for the case $m = \frac{1}{2}$). Namely, it was shown that for any $v \in \widetilde{H}^m(\Omega)$, the function $w_v(x, y)$ minimizing the weighted Dirichlet integral

$$\mathcal{E}_m(w) = \int\limits_0^\infty \int\limits_{\mathbb{R}^n} y^{1-2m} |\nabla w(x, y)|^2\, dxdy$$

over the set

$$\mathcal{W}(v) = \Big\{ w(x, y)\, :\, \mathcal{E}_m(w) < \infty\;,\quad w\big|_{y=0} = v \Big\},$$

satisfies

$$\int\limits_{\mathbb{R}^n} |\,(-\Delta)_N^{\frac{m}{2}} v|^2\, dx = c_m\, \mathcal{E}_m(w_v), \tag{2.4}$$

where the constant c_m depends only on m.

For any fixed $u \in \widetilde{H}^m(\Omega)$ find $w_u \in \mathcal{W}(u)$ and $w_{|u|} \in \mathcal{W}(|u|)$. Then clearly $|w_u| \in \mathcal{W}(|u|)$ and therefore $\mathcal{E}_m(w_{|u|}) \le \mathcal{E}_m(|w_u|) = \mathcal{E}_m(w_u)$. Thus (2.2) holds, thanks to (2.4).

Now assume that u changes sign. The function $w_{|u|}(x,y)$ is the unique solution of the boundary value problem

$$-\mathrm{div}(y^{1-2m}\nabla w) = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+; \qquad w\big|_{y=0} = |u| \tag{2.5}$$

with finite energy. Hence $w_{|u|}$ is analytic in $\mathbb{R}^n \times \mathbb{R}_+$. Since w_u changes sign then $|w_u|$ can not solve (2.5), that implies $\mathcal{E}_m(|w_u|) > \mathcal{E}_m(w_{|u|})$. Hence the strict inequality holds in (2.2), that concludes the proof for the Dirichlet Laplacian.

To check (2.3) one has to use, instead of [2], the characterization of the Navier fractional Laplacian given (among some other fractional operators) in [19]. Namely, for any $v \in \widetilde{H}^m(\Omega)$, the function $w_v^N(x,y)$ minimizing $\mathcal{E}_m(w)$ over the set

$$\mathcal{W}^N(v) = \Big\{ w \in \mathcal{W}(v) \, : \, \operatorname{supp} w(\,\cdot\,,y) \subseteq \overline{\Omega} \quad \text{for any } y > 0 \Big\},$$

satisfies

$$\int\limits_{\Omega} |\,(-\Delta)_N^{\frac{m}{2}} v|^2 \, dx = c_m \, \mathcal{E}_m(w_v).$$

The rest of the proof runs as in the Dirichlet case. We omit details. $\square$

Remark 1 *Here we deal with maximum principles for the operators $(-\Delta)_D^m$ and $(-\Delta)_N^m$, $m \in (0,1)$.*

Let $u \in \widetilde{H}^m(\Omega)$, and let $f = (-\Delta)_D^m u \in (\widetilde{H}^m(\Omega))'$ be a nonnegative and nontrivial distribution. Then it is well known that $u \ge 0$ in Ω. This is actually a simple corollary to Theorem 3. The function u is characterized variationally as the unique minimizer of the energy functional

$$J(v) = \int\limits_{\mathbb{R}^n} |\,(-\Delta)_D^{\frac{m}{2}} v|^2 \, dx - 2\,\langle f, v\rangle$$

on $\widetilde{H}^m(\Omega)$. We have $J(|u|) \le J(u)$ by Theorem 3. This implies $u = |u| \ge 0$, as desired, by the uniqueness of the minimizer.

By the same reason, if $u \in \widetilde{H}^m(\Omega)$ and $(-\Delta)_N^m u = f \ge 0$ then $u \ge 0$ in Ω.

5. We conclude this preliminary section by recalling a well known fact already mentioned in the Introduction.

Proposition 3 *Let $p > 1$, $m > 0$, $n > 2mp$. Then for any $f \in L^p(\mathbb{R}^n)$, problem*

$$(-\Delta)^m_D U = f; \qquad U \in \mathcal{D}^{2m,p}(\mathbb{R}^n)$$

has a unique solution. If in addition $f \neq 0$ is nonnegative, then $U > 0$ in $\mathbb{R}^n$.

Proof. Up to a multiplicative constant, the unique solution U is explicitly given by $|x|^{2m-n} \star f$. The statement readily follows. $\square$

3 Proof of Theorems 1 and 2

Since $\widetilde{H}^m(\Omega) \subseteq \widetilde{H}^m_N(\Omega)$, then clearly

$$\mathcal{S}^N_m = \inf_{\substack{u \in \widetilde{H}^m_N(\Omega) \\ u \neq 0}} \frac{Q^N_m[u]}{\|u\|^2_{2^*_m}} \leq \inf_{\substack{u \in \widetilde{H}^m(\Omega) \\ u \neq 0}} \frac{Q^N_m[u]}{\|u\|^2_{2^*_m}}.$$

Hence, $\mathcal{S}^N_m \leq \mathcal{S}_m$ by Proposition 1, if $2k - 1 \leq m \leq 2k$, $k \in \mathbb{N}$, and by Proposition 2, otherwise. By the same reason, $\mathcal{H}^N_m \leq \mathcal{H}_m$. Thus, it suffices to prove the opposite inequalities $\mathcal{S}^N_m \geq \mathcal{S}_m$ and $\mathcal{H}^N_m \geq \mathcal{H}_m$.

Fix any nontrivial $u \in \widetilde{H}^m_N(\Omega)$ and extend it by the null function. To conclude the proof, it is sufficient to construct a function $U \in \mathcal{D}^m(\mathbb{R}^n)$ such that

$$U \geq |u| \quad \text{a.e. in } \mathbb{R}^n; \tag{3.1}$$

$$\int\limits_{\mathbb{R}^n} |(-\Delta)^{\frac{m}{2}}_D U|^2 \, dx \leq \int\limits_{\Omega} |(-\Delta)^{\frac{m}{2}}_N u|^2 \, dx. \tag{3.2}$$

We have to distinguish between two cases.

1. <u>Case $2k + 1 < m \leq 2k + 2$, for some $k \in \mathbb{N}_0$.</u>

We use Proposition 3 to fix the unique positive solution U of

$$(-\Delta)^{\frac{m}{2}}_D U = \chi_\Omega |(-\Delta)^{\frac{m}{2}}_N u|; \qquad U \in \mathcal{D}^m(\mathbb{R}^n),$$

where $\chi_\Omega|\,(-\Delta)_N^{\frac{m}{2}}u|$ denotes the null extension of the function $|\,(-\Delta)_N^{\frac{m}{2}}u| \in L^2(\Omega)$. Since (3.2) trivially holds, we only have to check (3.1), that is the trickiest step in the whole proof.

It is convenient to write

$$\frac{m}{2} = k + \alpha \ , \quad \frac{1}{2} < \alpha \le 1.$$

Since $u \in \widetilde{H}_N^m(\Omega)$, then for any integer $\nu = 0, \cdots, k$ the function $u_\nu := (-\Delta)^\nu u$ belongs to $H_0^1(\Omega)$, compare with Lemma 1. In addition we know that $u_k \in \widetilde{H}_N^{2\alpha}(\Omega)$, that implies

$$u_k \in H_0^1(\Omega) \ , \quad (-\Delta)_N^\alpha u_k \in L^2(\Omega)\,.$$

We introduce the solutions $\widetilde{w}$, w to

$$\begin{aligned}
(-\Delta)_N^\alpha \widetilde{w} &= |\,(-\Delta)_N^\alpha u_k|; \qquad \widetilde{w} \in \widetilde{H}^\alpha(\Omega); \\
(-\Delta)_D^\alpha w &= |\,(-\Delta)_N^\alpha u_k|; \qquad w \in \widetilde{H}^\alpha(\Omega)\,.
\end{aligned}$$

We claim that

$$w \ge \widetilde{w} \ge |u_k| \quad \text{a.e. in } \ \Omega. \tag{3.3}$$

The fact that $\widetilde{w} \ge |u_k|$ readily follows from the maximum principle, see Remark 1 or [3, Lemma 2.5]. Also by the maximum principle w is nonnegative, and hence by [15, Theorem 1] we have $(-\Delta)_N^\alpha w \ge (-\Delta)_D^\alpha w$ in the distributional sense on Ω. Therefore,

$$(-\Delta)_N^\alpha (w - \widetilde{w}) \ge (-\Delta)_D^\alpha w - (-\Delta)_N^\alpha \widetilde{w} = 0,$$

and the maximum principle applies again to get (3.3).

Now we decompose $U \in \mathcal{D}^m(\mathbb{R}^n)$ in the same way as we did for u. Namely, we define $U_\nu = (-\Delta)^\nu U$ for any integer $\nu = 0, \cdots, k$, and notice that

$$(-\Delta)_D^{\frac{m}{2}-\nu} U_\nu = \chi_\Omega|\,(-\Delta)_N^{\frac{m}{2}}u|\ , \qquad U_\nu \in \mathcal{D}^{m-2\nu}(\mathbb{R}^n).$$

By Proposition 3, $U_\nu > 0$ on $\mathbb{R}^n$. In particular, the function $U_k \in \mathcal{D}^{2\alpha}(\mathbb{R}^n)$ solves

$$(-\Delta)_D^\alpha U_k = (-\Delta)_D^\alpha w \quad \text{in } \ \Omega; \qquad U_k > 0 = w \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}.$$

Therefore $U_k \ge w$ on Ω, and we have by (3.3)

$$U_k \ge |u_k| \quad \text{a.e. in } \ \Omega. \tag{3.4}$$

If $k = 0$ then we are done. If $k \geq 1$ then (3.4) is equivalent to

$$-\Delta U_{k-1} \geq |-\Delta u_{k-1}| \quad \text{a.e. in } \Omega,$$

that readily implies $U_{k-1} \geq |u_{k-1}|$ on Ω, as $U_{k-1} > 0$ on $\mathbb{R}^n$ and $u_{k-1} \equiv 0$ on $\mathbb{R}^n \setminus \overline{\Omega}$. Repeating the same argument we arrive at (3.1), and the proof is complete.

2. Case $2k < m \leq 2k+1$, for some $k \in \mathbb{N}_0$.
Now we write

$$\frac{m}{2} = k + \alpha \ , \quad 0 < \alpha \leq \frac{1}{2}.$$

From $u \in \widetilde{H}^m_N(\Omega)$ we infer that $(-\Delta)^k u \in \widetilde{H}^{2\alpha}(\Omega)$ by Lemma 1. Since $2\alpha \in (0,1]$, then also $|(-\Delta)^k u| \in \widetilde{H}^{2\alpha}(\Omega)$. By Sobolev embedding, $|(-\Delta)^k u| \in L^{2^*_{2\alpha}}(\Omega)$.

Notice that $n > 2k \cdot 2^*_{2\alpha}$. Therefore we can apply Proposition 3 with $m = k$ and $p = 2^*_{2\alpha}$ to find the unique positive solution U to

$$(-\Delta)^k U = |(-\Delta)^k u|; \qquad U \in \mathcal{D}^{2k, 2^*_{2\alpha}}(\mathbb{R}^n)\,.$$

Since $(2^*_{2\alpha})^*_{2k} = 2^*_m$, the Sobolev embedding theorem gives $U \in L^{2^*_m}(\mathbb{R}^n)$. Moreover, from $(-\Delta)^k U \in \mathcal{D}^{2\alpha}(\mathbb{R}^n)$ we infer that $(-\Delta)^{\frac{m}{2}}_D U \in L^2(\mathbb{R}^n)$, that is, $U \in \mathcal{D}^m(\mathbb{R}^n)$.

The proof of (3.1) runs now in the same way as in the case 1, and is even more simple since we only have to handle Laplacians of integer orders.

To check (3.2), we write

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} |\,(-\Delta)^{\frac{m}{2}}_D U|^2\,dx &= \int\limits_{\mathbb{R}^n} |\,(-\Delta)^{\alpha}_D \big(|(-\Delta)^k u|\big)|^2\,dx \\
&\leq \int\limits_{\mathbb{R}^n} |\,(-\Delta)^{\alpha}_D ((-\Delta)^k u)|^2\,dx \\
&\leq \int\limits_{\Omega} |\,(-\Delta)^{\alpha}_N ((-\Delta)^k u)|^2\,dx = \int\limits_{\Omega} |\,(-\Delta)^{\frac{m}{2}}_N u)|^2\,dx\,.
\end{aligned}$$

Here the first inequality holds by Theorem 3, the second one follows from (2.1) for $2\alpha = 1$ and from Proposition 1 for $2\alpha \in (0,1)$.

Thus, Theorems 1 and 2 are completely proved. $\square$

Remark 2 (Non-Hilbertian case) *Let $m \in \mathbb{N}$, and let $1 < p < \frac{n}{m}$. With minor modifications, one gets an alternative proof of [7, Theorems 1 and 2] concerning the Navier-Sobolev and Navier-Hardy constants for the space $W_N^{m,p}(\Omega)$. Best constants in weighted Sobolev inequalities can be included as well, see [14] for $m = 2$.*

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